Higher-Derivative Gravity with Non-minimally Coupled p-form

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Talk is based on works in collaboration with

- Xing-Hui Feng, arXiv:1512.09153
- Wei-Jian Geng, arXiv:1511.03681
- Xing-Hui Feng, Hai-Shan Liu and Chris Pope, arXiv:1509.07142; arXiv:1512.02659

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Outline

- Motivation
- Construction of a general theory involving a p-form
- $p = 1$ field strength: Horndeski gravity
- $p = 2$ field strength
- $p = 1$ vector potential
- Conclusions

Motivation

Basic assumptions in Einstein's formulation of Gravity

- General coordinate covariance or invariance;
- The metric is the fundamental field, describing spacetime geometry;
- Two derivatives;
- Matter couples minimally.

Consequently

$$\mathcal{L} = \sqrt{-g}\big(R + L^{\mathsf{mat}}\big) \qquad \rightarrow \qquad -G_{\mu\nu} + T^{\mathsf{mat}}_{\mu\nu} = 0\,,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$.

(It is rather unnatural not to include the cosmological constant.)

A natural generalization: higher derivatives

Since the metric is dimensionless, higher-derivative gravity requires a fundamental constant with the dimension of length. Such a constant indeed exists: it is the Planck length $\ell_p = \sqrt{G_N}$. (It would be very unnatural to consider higher-derivative theories in QFT, for lacking of such a fundamental constant.)

Planck length ℓ_p in $D = 4$ is very small, but $\widehat{\ell}_p$ in higher dimensions may not be, since

$$\widehat{\ell}_p^{\frac{1}{2}(D-2)} = L^{\frac{1}{2}(D-4)}\,\ell_p\,,$$

where L is the length scale of internal dimensions.

Pros and cons in higher-derivative gravities

Higher-derivative gravity: Renormalizable, but contains ghost-like massive spin-2 modes (K.S.Stelle,70s)

There can exist some critical points of the parameter space such that the ghost-like massive graviton modes become log modes. This may lead to possible quantum gravity in $D = 3$ (Li,Song,Strominger). The critical gravity in $D = 4$ may be more problematic (Lu, Pope)

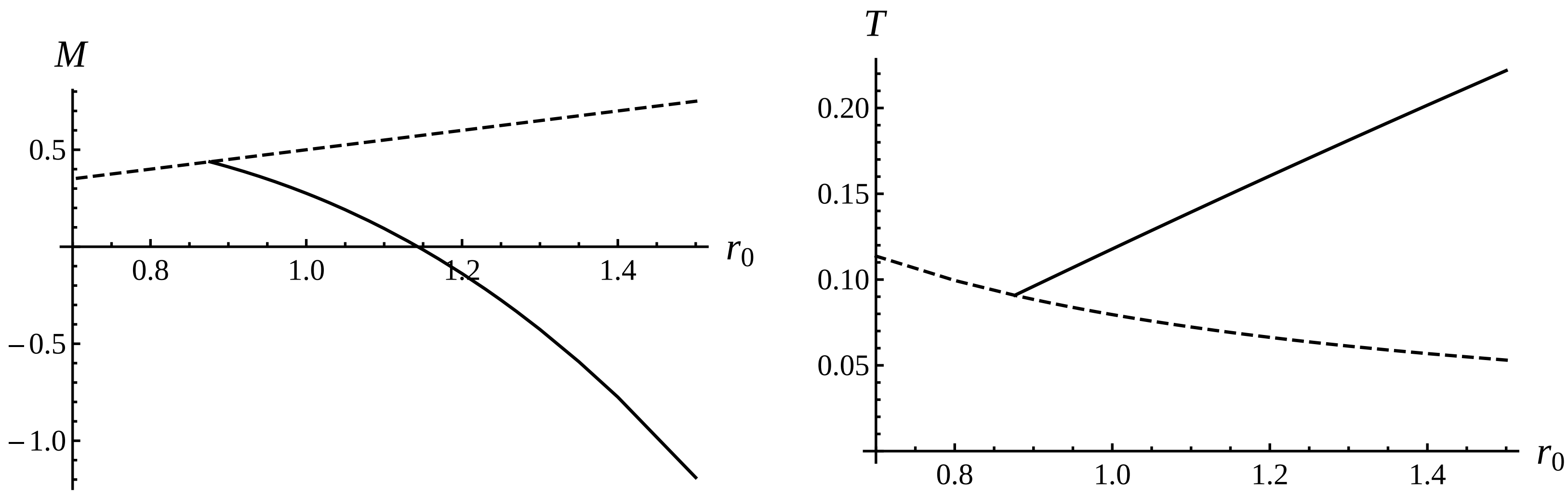
New black holes in higher-derivative gravity

Einstein gravity extended with quadratic curvature invariants admit the Schwarzschild black hole without any modification (only) in four dimensions.

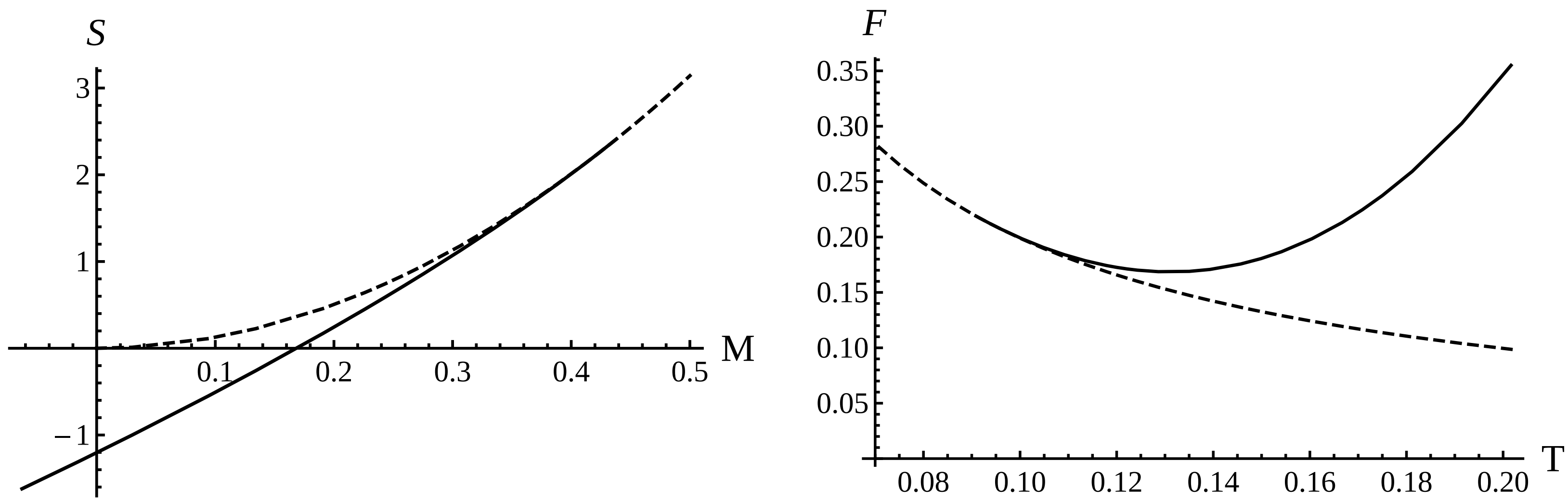
Does the theory, considered by Stelle in 70s, contain further black holes? One might expect a no-go theorem.

Recently it was established a new asymptotically-flat black hole that is disjoint from the usual Schwarzschild one exists indeed. It contains Yukawa falloff $\mu e^{-m_2 r}/r$, indicating condensation of the massive graviton. (massive spin-2 hair.) [Lu,Perkins, Pope, Stelle]

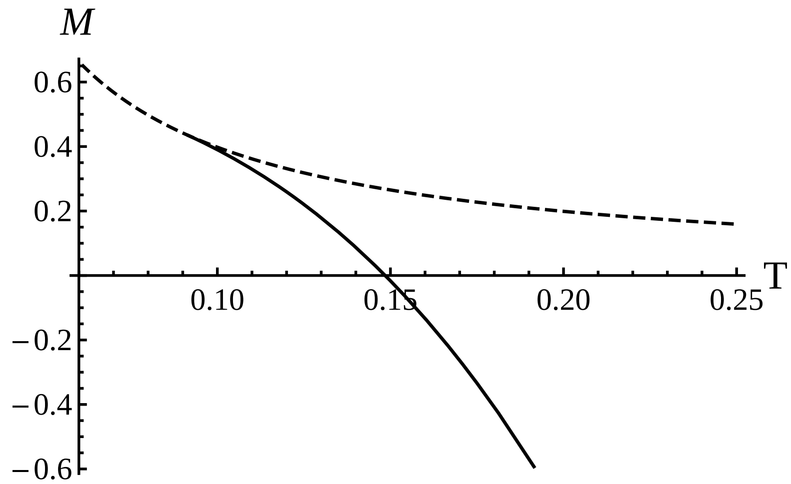
Properties of the new black hole



The masses (left plot) and temperatures (right plot) of the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes as a function of the horizon radius r_0.



The first plot shows the entropy as a function of mass, and the second shows the free energy $F = M - TS$ as a function of T, for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.



The mass M as the function of temperature T.

It is clear that we have

$$C_{\text{new}} < C_{\text{Schw}} < 0\,.$$

All fields have hair! while all "objects" cannot move out of the horizon of a black hole, the tentacles of a field can seep out.

Far more number of new black holes are expected to exist in generic higher-derivative gravity, even though exact solutions are rare.

Avoiding ghosts: Gauss-Bonnet or Lovelock terms: total derivatives in $D = 4$: these terms are naturally existing in higher dimensions.

Higher derivative gravities

String theories predict that two-derivative supergravities are their low energy effective theories. Higher-order α' and/or string loop corrections involve higher derivatives in the metric. E.g. the α' correction of $\mathcal{N} = 1$, $D = 10$ supergravity contains de Roo, Bergshoeff

$$\alpha' R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \,.$$

Such a term makes sense only perturbatively, since when treated on its own, the theory has inevitable ghosts.

While many aspects of the supergravities can be discussed with small α', non-trivial application requires that the higher-order term contributes none infinitesimally. E.g. Cosmology, AdS/CFT correspondence, etc.

Gauss-Bonnet and Lovelock gravities

In string theory, one can perform picture change or field redefinition a la

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = g_{\mu\nu} + c_1 \alpha' R_{\mu\nu} + c_2 \alpha' R\, g_{\mu\nu}\,,$$

such that the Riemann-squared term becomes the Gauss-Bonnet term

$$\alpha' R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \quad \to \quad \alpha' (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2)\,.$$

The equations of motion of the Gauss-Bonnet term remain second order and hence can be ghost free.

One then typically appeals to the hand-waving argument that in the enormous landscape of string vacua, there are cases where the Gauss-Bonnet term dominates and the low-energy effective theory can be treated on its own.

In fact, the Ricci scalar, Gauss-Bonnet term are both Euler integrands; the former is of two derivatives whilst the latter is of four-derivatives. There are an infinite series of such Euler integrands, giving rise to the general Lovelock gravities.

Including matter

Matter in two-derivatives supergravities are in general minimally coupled, at least in the Einstein frame. However, when higher-order terms involved, matters are non-minimally coupled. We expect that there may exist ghost-free combination at some finite order by appropriate field redefinition, as in the case of Lovelock gravities.

Also the dimensional reduction of Lovelock gravity will give rise to lower-dimensional gravities with non-minimal couplings.

The natural field content of the bosonic sector in the low-energy effective theory consists the metric and various p-forms.

As a toy model, we consider higher-derivative gravities with one non-minimally coupled p-form. The goal is to construct a higher-order theory whose field equations of motion remain nevertheless second order.

Generalizations of Einstein gravity

- General coordinate covariance or invariance.
- The metric is the fundamental field, together with a p-form.
- nonlinear higher total derivatives using polynomial invariants constructed from the Riemann tensor and the p-form.
- Matter thus couples non-minimally.

Requirement: Field equations remain second order in derivatives, analogous to Lovelock gravities.

Lovelock gravity

We start with an Euler integrand of $(2k)$'th order

$$E^{(k)} = \frac{1}{2^k}\delta^{c_1 d_1 \cdots c_k d_k}_{a_1 b_1 \cdots a_k b_k} R^{a_1 b_1}_{c_1 d_1} \cdots R^{a_k b_k}_{c_k d_k}\,,$$

where R^{ab}_{cd} denotes the Riemann tensor $R^{ab}{}_{cd}$ and

$$\delta^{\beta_1 \cdots \beta_s}_{\alpha_1 \cdots \alpha_s} = s!\,\delta^{\beta_1}_{[\alpha_1} \cdots \delta^{\beta_s}_{\alpha_s]}\,.$$

This implies that the Euler integrands can also be expressed as

$$E^{(k)} = \frac{(2k)!}{2^k} R^{[a_1 b_1}_{a_1 b_1} \cdots R^{a_k b_k]}_{a_k b_k}\,.$$

The low-lying examples are

$$E^{(0)} = 1\,, \quad E^{(1)} = R\,, \quad E^{(2)} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,, \quad etc.$$

The term $\sqrt{-g}E^{(k)}$ in the Lagrangian contributes

$$E^{(k)\,\nu}_{\mu} = -\frac{1}{2^{k+1}}\delta^{a_1 b_1 \cdots a_k b_k\,\nu}_{c_1 d_1 \cdots c_k d_k\,\mu} R^{a_1 b_1}_{c_1 d_1} \cdots R^{a_k b_k}_{c_k d_k}$$

to the Einstein's equation of motion.

A striking property is that no Riemann-tensor factor acquires any derivative in the equations of motion. This is a consequence of the fact that the variation of the Riemann tensor, namely

$$\delta R^{\mu}{}_{\nu\rho\sigma} = \nabla_{\rho}\delta\Gamma^{\mu}_{\sigma\nu} - \nabla_{\sigma}\delta\Gamma^{\mu}_{\rho\nu}$$

yields a total derivative term in the Lagrangian for the polynomial combinations of the Euler integrands. This is largely due to the Bianchi identity of the Riemann tensor, namely

$$\nabla_{[\alpha}R^{\mu\nu}_{\rho\sigma]} = 0 = \nabla^{[\beta}R^{\mu\nu]}_{\rho\sigma}\,.$$

With 2-form field strength

Consider a Maxwell field A whose field strength is $F = dA$. Introduce

$$Z^{ab}_{cd} = F^{ab} F_{cd}\,.$$

Owing to the Bianchi identity of the Maxwell field, namely

$$\nabla_{[\alpha} F_{\rho\sigma]} = 0 = \nabla^{[\beta} F^{\mu\nu]}\,,$$

the Z tensor satisfies the property

$$\nabla_{[\alpha} \nabla^{[\beta} Z^{\mu\nu]}_{\rho\sigma]} = \nabla_{[\alpha} F^{[\mu\nu} \nabla^{\beta]} F_{\rho\sigma]} + 2 F^{[\mu\nu}\, R^{\beta]}{}_{[\rho\sigma}{}^{\lambda}\, F_{\alpha]\lambda}\,.$$

In other words, although each term involves a total of four derivatives, both A_μ and $g_{\mu\nu}$ have at most two derivatives acting directly. This property is crucial our construction.

With these preliminaries, we consider polynomial invariants of the tensor R^{ab}_{cd} and Z^{ab}_{cd} analogous to the Euler integrants, namely

$$L^{(m,n)} = \frac{(2(m+n))!}{2^{m+n}}\, R^{[a_1 b_1}_{a_1 b_1} \cdots R^{a_m b_m}_{a_m b_m} Z^{\tilde{a}_1 \tilde{b}_1}_{\tilde{a}_1 \tilde{b}_1} \cdots Z^{\tilde{a}_m \tilde{b}_m]}_{\tilde{a}_m \tilde{b}_m}\,.$$

It is clear that when $n = 0$, the above gives rise to the Euler integrands, i.e.

$$L^{(k,0)} = E^{(k)}\,.$$

The Lagrangian for the general theory is then given by

$$\mathcal{L} = \sqrt{-g} \sum_{k=0} \sum_{m+n=k} \gamma_{mn} L^{(m,n)} \,,$$

where γ_{mn} are coupling constants. It is somewhat tedious but straightforward to verify that all the field equations remain second order in derivatives.

General p-form

The construction in the previous section can be easily generalized to general $(p-1)$-form potential $A_{(p-1)}$ whose p-form field strength is given by

$$F_{(p)} = dA_{(p-1)}\,, \qquad F_{a^1\cdots a^p} = p\,\nabla_{[a^1} A_{a^2\cdots a^p]}\,.$$

For simplicity of notations, we construct corresponding Z tensors

$$Z^{a^1\cdots a^p}_{b^1\cdots b^p} = F^{a^1\cdots a^p} F_{b^1\cdots b^p}\,.$$

The generalizing polynomial of the p-form to $L^{(m,n)}$ of the 2-form field strength is then given by

$$L^{(m,n),p} = \frac{(2m+pn)!}{2^m (p!)^n} R^{[a_1 b_1}_{[a_1 b_1} \cdots R^{a_m b_m}_{a_m b_m} Z^{a^1_1 \cdots a^p_1}_{a^1_1 \cdots a^p_1} \cdots Z^{a^1_n \cdots a^p_n]}_{a^1_n \cdots a^p_n]}\,,$$

When p is odd, we have $L^{(m,n),p} = 0$ for $n \geq 2$.

As p increases, the above construction is not unique. We shall not classify all possible ghost-free structures here.

$p = 1$, Horndeski gravity

When $p = 1$, the 1-form field strength of an axion-like scalar χ is $d\chi = \partial_\mu \chi dx^\mu$. A low-lying example of Einstein-Horndeski gravity is

$$I = \frac{1}{16\pi} \int d^n x \sqrt{-g}\, L\,, \qquad L = \kappa(R - 2\Lambda) - \tfrac{1}{2}(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\partial^\mu \chi\, \partial^\nu \chi\,.$$

This class of theories were constructed by Horndeski in the 70s. (Int.J.Theor.Phys.**10**,363 (1974).)

It was recently "rediscovered" in cosmology by covariantizing higher-derivative Galileon gravity. Galileon Gravity: Nicolis, Rattazzi and Trincherini, 0811.2197; Galileon/Horndeski relation: Deffayet and Steer, 1307.2450.

Two surprises

Static (AdS) black holes in Horndeski gravity were constructed (e.g. Anabalon, Cisterna and Oliva, 1312.3597.) (See also, Rinaldi, 1208.0103; Babichev, Charmousis, 1312.3204, Minamitsuji, 1312.3759, Sotiriou, Zhou, 1408.1698, Babichev, Charmousis, Hassaine, 1503.02545.)

Thermodynamical properties were recently analysed in our papers 1509.07142, 1512.02659, and we found two surprises

- The Wald entropy formula $S = -\frac{1}{8}\int d^{n-2}x\sqrt{h}\frac{\partial L}{\partial R^{abcd}}\epsilon^{ab}\epsilon^{cd}$ is not valid.
- The black hole quantum statistic relation $IT = M - TS$ is not valid, where I is the Euclidean action.

It is natural to expect that the mass of a black hole is $\frac{1}{2}\mu$ where μ is associated with the condensation of graviton $-g_{tt} = \cdots - \mu/r \cdots$. In either method, they will give a mass that is a complicated function of μ that involves some convoluted inverse of the hypergeometric function.

$\boxed{\text{Three surprises}}$

Going through the Wald formalism carefully, namely

$$\delta\mathcal{H} = \int \delta Q - i_\xi \Theta \tag{1}$$

we find that Wald obtained his famous entropy formula by evaluating the above formula on the horizon, assuming the second term has no contribution. This is no longer true for the black hole in Horndeski gravity, owing to the fact that $(\partial\chi)^2$ is non-vanishing on the horizon.

The completion of the first law requires an introduction of scalar charge on the horizon, namely $\sqrt{(\partial\chi)^2} \sim Q_\chi$ evaluated on the horizon, and the first law becomes $dM = TdS + \Phi_e dQ_e + \Phi_\chi^+ dQ_\chi^+$.

With applying the Wald formalism properly, we indeed find that $M = \frac{1}{2}\mu$.

Since one does not expect surprises in classical gravity, so it is worth checking the papers out.

Non-minimally coupled Maxwell field $p = 2$

Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big(R - 2\Lambda_0 - \tfrac{1}{4}F^2 + \gamma L^{(1,1)}\Big)\,,$$

where

$$L^{(1,1)} = \tfrac{1}{4}\delta^{cd\tilde{c}\tilde{d}}_{ab\tilde{a}\tilde{b}} R^{ab}_{cd} Z^{\tilde{a}\tilde{b}}_{\tilde{c}\tilde{d}} = RF^2 - 4R_{ab}F^{ac}F^b{}_c + R_{abcd}F^{ab}F^{cd}\,.$$

In other words, the theory is the Einstein-Maxwell theory with a cosmological constant, with an additional $L^{(1,1)}$ term. The Einstein equations of motion are

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} - \tfrac{1}{2}(F^2_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}F^2) + \gamma L^{(1,1)}_{(\mu\nu)} = 0\,,$$

where

$$\begin{aligned} L^{(1,1)}_{\mu\nu} \;&=\; -\tfrac{1}{2}g_{\mu\nu}L^{(1,1)} + \tfrac{1}{2}\delta^{cd\tilde{c}\tilde{d}}_{ab\tilde{a}\mu}\, R^{ab}_{cd}\, F_{\tilde{c}\tilde{d}}F^{\tilde{a}}{}_\nu + \tfrac{1}{4}\delta^{cd\tilde{c}\tilde{d}}_{a\mu\tilde{a}\tilde{b}}\, R^a{}_{\nu cd}\, Z^{\tilde{a}\tilde{b}}_{\tilde{c}\tilde{d}} \\ &\quad + \tfrac{1}{2}g_{c\mu}\delta^{cd\tilde{c}\tilde{d}}_{\nu b\tilde{a}\tilde{b}} \nabla^b\nabla_d\,(Z^{\tilde{a}\tilde{b}}_{\tilde{c}\tilde{d}})\,. \end{aligned}$$

The Maxwell equation is

$$\nabla_\mu \widehat{F}^{\mu\nu} = 0\,,\qquad \text{with} \qquad \widehat{F}^{\mu\nu} \equiv F^{\mu\nu} - \delta^{cd\mu\nu}_{ab\tilde{a}\tilde{b}} R^{ab}_{cd} F^{\tilde{a}\tilde{b}}\,.$$

Owing to the Bianchi identity of the Riemann tensor, the differential operator ∇_μ can only land on F, but not R.

Black holes in four dimensions

The general static black holes involve three parameters, the mass M, electric charge Q_e and magnetic charge Q_m. We find analytical solutions for the following parametrizations (1512.09153)

- Purely magnetic $Q_e = 0$;
- Purely electric $z = 2$ charged Lifshitz black hole $Q_m = 0$;
- Small Q_e and Q_m, up to quadratic order in Q_e^2 and Q_m^2;
- Small coupling γ, up to the linear order of γ, with general dyonic charge.

Black hole thermodynamics

- Wald entropy formula works.
- QSR does not. Maybe a general feature for these theories. Possible the case for all string theories, or how string theories evade this?

E.g. $z = 2$ Lifshitz black hole

When the parameters satisfy $8\gamma\Lambda_0 = 3$, we have

$$
\begin{aligned}
ds^2 &= -h dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{2,\epsilon}^2\,, \qquad A = \phi dt\,,\\
f &= g^2 r^2 + \epsilon - \frac{gq\sqrt{r^2+4\mu}}{2r}\,, \qquad h = (r^2+4\mu) f\,,\\
\phi &= g(r^2 - r_0^2)\,, \qquad \Lambda_0 = -\tfrac{1}{3} g^2\,.
\end{aligned}
$$

$$
\begin{aligned}
T &= \frac{\sqrt{f'(r_0) h'(r_0)}}{4\pi}\,, \qquad S = \pi r_0^2\,,\\
M &= \tfrac{1}{2} g q \mu\,, \qquad Q_e = \tfrac{1}{4} q\,, \qquad \Phi_e = -g(r_0^2 + 2\mu)\,.
\end{aligned}
$$

$$
dM = T dS + \Phi_e dQ_e\,, \qquad M = \tfrac{1}{2}(TS + \Phi_e Q_e)\,.
$$

Application in the AdS/CFT

Viscosity/entropy ratio

$$\frac{\eta}{S} = \frac{1}{4\pi}\frac{r_0^4 + 32\gamma Q_e^2}{r_0^4 + 32\gamma Q_m^2}\,, \qquad Q_e Q_m = 0\,.$$

The result was obtained without having to know the detail of the solutions and hence the parameters are universally applicable.

This is satisfactory since one of our motivation is to construct higher-derivative gravities where the coupling constants do not have to be small.

Einstein-vector theory

Let us consider the Einstein-Maxwell gravity

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{4}F^2)\,, \qquad F = dA\,.$$

we add a non-minimal coupling between gravity and the vector

$$\sqrt{-g}\,\gamma\, G_{\mu\nu} A^\mu A^\nu\,,$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and γ is some coupling constant. (1511.03681, with Wei-Jian Geng.)

The theory can be viewed as the gauging of the axion global shifting symmetry of Horndeski gravity: $\partial_\mu \chi \to \partial_\mu \chi + A_\mu \to A_\mu$.

(* This "gauging" can be done to any p-form field strength to become p-form potential. *)

We see immediately that the $U(1)$ symmetry is broken. So now the question is whether this term can be easily invalidated by an experiment or observation?

$\boxed{\text{Experiment test}}$

Owing to the fact that gravity is extremely weak compared to other forces, we typically ignore gravity in elementary particle physics, in our current experimental scale.

This term is unlikely to have any testable effect on LHC physics.

Observational test: Is light still light?

In the Minkowski vacuum or some more general backgrounds such as Schwarzschild or Kerr black holes, $G_{\mu\nu}$ vanishes. The linear fluctuations of the theory in these backgrounds consist only the massless graviton and photon, and hence the $U(1)$ gauge symmetry emerges. So we do see light as light at the linear level.

However, in our universe, $G_{\mu\nu} \neq 0$.

Observational effect, due to matter

In our current Universe, the contribution to the spacetime curvature due to electric-magnetic fields are negligible, and hence it can be viewed as a background with vanishing A. The matter energy-momentum tensor in the Einstein equation

$$G_{\mu\nu} = T^{\text{mat}}_{\mu\nu}\,,$$

has mainly three sources

- Baryon and lepton matter. Typically localized. For dust-like distributed matter, its direct interaction with light is far greater than this term.
- Dark energy: Could give a global mass to A, although not necessary. Extremely small anyway. Couloum's law remains effective long ranged.
- Dark matter: Can give a Lorentz violating and gauge-symmetry breaking, and should be observable.

$$\sqrt{-g}\,\gamma\,G_{\mu\nu}A^\mu A^\nu \to \sqrt{-g}\,\gamma\,T^{\text{DM}}_{\mu\nu}A^\mu A^\nu\,,$$

No observable effects within the scale of the solar system.

Two-derivative theory

We shall focus on the simpler theory where curvature appears linearly

$$\mathcal{L} = \sqrt{-g}\Big(R - 2\Lambda_0 - \tfrac{1}{4}F^2 - \tfrac{1}{2}\mu_0^2 A^2 + \beta R A^2 + \gamma G_{\mu\nu} A^\mu A^\nu\Big)\,.$$

Here $(\Lambda_0, \mu_0, \beta, \gamma)$ are constants.

Vacuum: (A)dS with

$$G_{\mu\nu} = -\Lambda_0 g_{\mu\nu}\,, \qquad\qquad A = 0\,,$$

Vacuum linear fluctuations consist only of a massless graviton and a Proca field with mass

$$\mu_{\rm eff}^2 = \mu_0^2 - \tfrac{4D}{D-2}\beta\Lambda_0 + 2\gamma\Lambda_0\,.$$

Becomes a Maxwell field if $\mu_{\rm eff} = 0$.

Vector Cosmology

FLRW ansatz

$$ds^2 = -dt^2 + a(t)^2(dx_1^2 + dx_2^2 + dx_3^2)\,, \qquad\qquad A = \phi(t)dt\,.$$

If A were a Maxwell field, this ansatz would be a pure gauge.

Cosmological application were also studied by Jimenez, Maroto

de Sitter bounce

Redefine the parameters

$$\mu_0^2 = 12\beta\,\mu^2\nu\,,\qquad \gamma = \frac{2\beta(2\nu-1)}{\nu}\,.$$

The general solution is given by

$$a = [\cosh(\mu t)]^\nu\,,\qquad \phi^2 = \sinh(\mu t)[\cosh(\mu t)]^{1-3\nu}\,\psi\,,$$

where

$$\dot\psi = \frac{\mu\nu}{\beta}[\cosh(\mu t)]^{3\nu-2} - \frac{\Lambda_0[\cosh(\mu t)]^{3\nu}}{3\beta\mu\nu\,[\sinh(\mu t)]^2}\,.$$

This can be solved in terms of hypergeometric functions, given by

$$\begin{aligned}\psi &= \frac{\nu}{\beta}\Big(\psi_0 + {}_2F_1[\tfrac12,-\tfrac32(\nu-1);\tfrac32;-\sinh^2(\mu t)]\sinh(\mu t)\\ &\quad + \frac{\Lambda_0}{\Lambda_{\rm eff}\sinh\mu t}\,{}_2F_1[-\tfrac12,-\tfrac12(3\nu-1);\tfrac12;-\sinh^2(\mu t)]\Big)\,,\end{aligned}$$

where $\Lambda_{\rm eff} = 3\mu^2\nu^2$ and ψ_0 is an integration constant, which we shall set 0. The metric function a clearly describes a bounce at $t=0$.

The reality condition for ϕ for all the comoving time range $(-\infty,\infty)$ gives some restrictions on Λ_0 and ψ_0. We shall focus on the case with $\beta\nu > 0$.

For $\nu > \frac{2}{3}$, in the asymptotic $t \to \pm\infty$ region, the function ϕ approaches a constant, given by

$$\phi^2 \to \frac{\nu}{(3\nu-2)\beta}\Big(1 - \frac{\Lambda_0}{\Lambda_{\rm eff}}\Big)\,.$$

Thus the full reality condition requires that $0 \le \Lambda_0 \le \Lambda_{\rm eff}$. In one limit, $\Lambda_0 = 0$, we have

$$\phi^2 = \frac{\nu}{\beta}[\cosh(\mu t)]^{1-3\nu}\sinh^2(\mu t)\,{}_2F_1[\tfrac{1}{2}, -\tfrac{3}{2}(\nu-1); \tfrac{3}{2}; -\sinh^2(\mu t)]\,.$$

This de Sitter bounce is generated by "dark energy" without bared cosmological constant. Although the hypergeometric function is already rather straightforward, we present the simplest $\nu = 1$ solution in which the hypergeometric function becomes identity.

The $\nu = 1$ bouncing universe with $\Lambda_0 = 0$ is

$$ds^2 = -dt^2 + \cosh^2(\mu t)(dx_1^2 + dx_2^2 + dx_3^2)\,, \qquad A = \frac{\tanh \mu t}{\sqrt{\beta}}\, dt\,.$$

In the other limit with $\Lambda_0 = \Lambda_{\rm eff}$, we have

$$\phi^2 = \frac{\nu}{\beta}[\cosh(\mu t)]^{1-3\nu}\, {}_2F_1[\tfrac{1}{2}, -\tfrac{3}{2}(\nu-1); \tfrac{1}{2}; -\sinh^2(\mu t)]\,.$$

The $\nu = 1$ solution is given by

$$ds^2 = -dt^2 + \cosh^2(\mu t)(dx_1^2 + dx_2^2 + dx_3^2)\,, \quad A = \frac{1}{\sqrt{\beta}\, \cosh \mu t}\, dt\,.$$

The $\Lambda_{\rm eff} = \Lambda_0$ solutions are particularly interesting, since as $t \to \pm\infty$, we have $\phi \to 0$. Furthermore, it turns out in this case, the effective mass $\mu_{\rm eff}$ of the "photon" field, vanishes precisely.

Thus the $\Lambda_0 = \Lambda_{\rm eff}$ solutions describe the bounce between two de Sitter vacua with $A = 0$, whose linear spectrum contains precisely one graviton and one photon.

Resolving cosmic singularity?

A concrete example of $\Lambda_0 = \Lambda_{\text{eff}} = 3\mu^2\nu^2$, and $\nu = 1$.

$$ds^2 = -dt^2 + \cosh^2(\mu t)(dx_1^2 + dx_2^2 + dx_3^2)\,, \qquad A = \frac{1}{\sqrt{\beta}\,\cosh\mu t}\,dt\,.$$

$$\mu_0^2 = 4\beta\mu^2\,, \qquad \gamma = 2\beta\,, \qquad \Lambda_0 = 3\mu^2\,.$$

Static solutions Black holes

Let us consider

$$\mathcal{L}_1 = \sqrt{-g}\Big(R - \tfrac{1}{4}F^2 + \gamma G_{\mu\nu} A^\mu A^\nu\Big)\,.$$

Schwarzschild and Kerr black holes are solutions, and hence Newtonian gravity can be recovered. But we find a new asymptotically-flat black hole for $\gamma = \frac{1}{4}$, namely

$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2\,, \qquad A = 2\sqrt{2}\,f\,dt\,, \qquad f = 1 - \sqrt{\frac{r_0}{r}}\,.$$

Such a black hole cannot be ruled out since Newtonian gravity is not application in the scale of a galaxy. For $\gamma \neq \frac{1}{4}$, the leading falloff is the standard $1/r$.

Not clear if there is any significance of this solution.

Wormholes

The theory

$$\mathcal{L}_2 = \sqrt{-g}\Big(R - 2\Lambda_0 - \tfrac{1}{4}F^2 + \gamma(G_{\mu\nu} + \Lambda_0 g_{\mu\nu})A^\mu A^\nu\Big)\,.$$

Solutions (Asymptotically (A)dS)

$$ds^2 = -h\,dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{2,\epsilon}^2\,, \qquad A = \phi dt\,.$$

$$f = -\tfrac{1}{3}\Lambda_0 r^2 + \epsilon - \tfrac{m}{r}\,, \quad h = h_0 + \tfrac{1}{2}\Big(1 + \sqrt{1 + \tfrac{4h_0}{f}}\Big) f\,, \quad \phi = \sqrt{-\tfrac{2h_0}{\gamma}}\,.$$

Asymptotically flat $\Lambda_0 = 0$ and $\epsilon = 1$:

$$h = \begin{cases} h_0 + \tfrac{1}{2}(h_0 - 1)^2\Big(1 + \sqrt{1 + \tfrac{4h_0}{(h_0-1)^2 f}}\Big) f, & 0 < h_0 < 1; \\ 1, & h_0 = 1; \\ h_0 + \tfrac{1}{2}(h_0 - 1)^2\Big(1 - \sqrt{1 + \tfrac{4h_0}{(h_0-1)^2 f}}\Big) f, & h_0 > 1\,. \end{cases}$$

$$f = 1 - \frac{m}{r}\,, \qquad \phi = \phi_0 \equiv \sqrt{-\frac{2h_0}{\gamma}}\,, \qquad \gamma < 0\,.$$

A superpotential method

Consider the ansatz:

$$ds^2 = a^2 b^4\, d\rho^2 - a^2 dt^2 + b^2 d\Omega_2^2\,, \qquad A = \phi\, dt\,,$$

where (a, b, ϕ) are functions of the radial coordinate ρ. We find that the effective one-dimensional Lagrangian is $L = T - V$ where the kinetic T and potential V energies are

$$\begin{aligned} T &= \left(2 - \frac{\gamma\phi^2}{a^2}\right)\frac{a'b'}{ab} + \left(1 + \frac{\gamma\phi^2}{a^2}\right)\frac{b'^2}{b^2} + \frac{2\gamma\phi\phi' b'}{a^2 b} + \frac{\phi'^2}{4a^2}\,, \\ V &= \frac{1}{4}b^2(-4a^2 + 4\Lambda_0 a^2 b^2 - 2\gamma\phi^2 + 2\gamma\Lambda_0 b^2\phi^2)\,. \end{aligned}$$

The superpotential method is first to treat the kinetic term T as some one-dimensional σ-model

$$T = \tfrac{1}{2} g_{ij} (X^i)'(X^j)'\,.$$

If the potential V can be expressed in terms of a superpotential W as

$$V = -\tfrac{1}{2} g^{ij} \tfrac{dW}{dX^i}\tfrac{dW}{dX^j}\,,$$

the Lagrangian then admits special solutions that satisfy the first-order equations

$$(X^i)' = g^{ij}\frac{\partial W}{\partial X^j}\,.$$

For the static ansatz, we find that a superpotential indeed exists, given by

$$W=\frac{1}{a}(2a^2+\gamma\phi^2)\sqrt{b(b\,\epsilon-\tfrac{1}{3}\Lambda_0 b^3-m)}\,,$$

where m is an arbitrary constant, which turns out to be the mass of the solution. The resulting first-order equations are

$$a'=\frac{ab(3m-\Lambda_0 b^3)(2a^2+\gamma\phi^2)^2}{6W(2a^2-\gamma\phi^2)}\,,\qquad b'=-\frac{b^2(3m-\Lambda_0 b^3)(2a^2+\gamma\phi^2)^2}{3W}\,,$$

These two first-order equations can be solved easily and we find that the solutions are the wormholes that we saw earlier.

The wormholes arise as solutions of first-order system is suggestive that they are stable against perturbation.

Lifshitz black holes

$$ds^2 = \ell^2\Big(-r^{2z} f dt^2 + \frac{dr^2}{r^2} + r^2(dx_1^2 + dx_2^2)\Big)\,, \qquad A = q r^z\, \ell f dt\,.$$

$$f = 1 - \Big(\frac{r_0}{r}\Big)^{1+\frac{1}{2}z}\,,$$

$$\begin{aligned} q^2 &= \frac{4(z-1)(4-z)(3z+2)}{z(z+2)(z+10)}\,, \quad \ell^2 = \frac{z(z+2)(2+5z+4z^2-2z^3)}{\mu_0^2(z-1)(4-z)(3z+2)}\,,\\ \gamma &= -\frac{(z+2)(8-8z+5z^2)}{2(z-1)(z-4)(3z+2)}\,, \quad \beta = -\frac{z^2-z+2}{2(z-1)(3z+2)}\,,\\ \Lambda_0 &= -\frac{z(z+2)(3+5z+z^2)q^2}{4(z-1)(4-z)(3z+2)\ell^2}\,. \end{aligned}$$

Charged Lifshitz black holes

With an additional Maxwell field $\mathcal{A}$, we find a charged $z = 6$ Lifshitz black hole. We present the solution in four dimensions

$$\begin{aligned} ds^2 &= \ell^2\Big(-r^{12} f dt^2 + \frac{dr^2}{r^2 f} + r^2(dx_1^2 + dx_2^2)\Big)\,, A = \sqrt{\tfrac{5}{8(1+10\beta)}}\, f\, r^6\, \ell dt\,, \\ \mathcal{A} &= (\psi_0 + Q r^4)\, \ell dt\,, \qquad f = 1 - \frac{4(1+10\beta)Q^2}{(4+25\beta)r^4}\,, \\ \Lambda_0 &= \frac{(7+40\beta)\mu_0^2}{256\beta(1+10\beta)}\,, \qquad \gamma = -2 - 30\beta\,, \qquad \ell^2 = -\frac{384\beta}{\mu_0^2}\,. \end{aligned}$$

where the parameter $-\frac{1}{10} < \beta \le \infty$, with the following constraint

$$\begin{cases} -\frac{1}{10} < \beta < 0\,, & \mu_0^2 > 0\,; \\ \beta = 0\,, & \mu_0^2 = 0\,; \\ \beta > 0\,, & \mu_0^2 < 0\,. \end{cases}$$

Randall-Sundrum domain walls

$$ds^2 = dr^2 + a(r)^2(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)\,, \qquad A = \phi(r)\,dr\,.$$

We rewrite the parameters in terms of (μ, ν) given by

$$\mu_0^2 = 16\beta\,\mu^2\nu\,, \qquad \gamma = \frac{2\beta(5\nu+2)}{3\nu}\,.$$

The domain-wall solutions are given by

$$\begin{aligned} a &= \frac{1}{\cosh^\nu(\mu r)}\,, \qquad \phi^2 = \frac{3\nu}{2\beta}[\cosh(\mu r)]^{1+4\nu}\psi\,, \\ \psi &= \psi_0 \sinh(\mu r) + \sinh^2(\mu r)\,{}_2F_1[\tfrac{1}{2}, \tfrac{3}{2}+2\nu; \tfrac{3}{2}; -\sinh^2(\mu r)] \\ &\quad + \frac{\Lambda_0}{\Lambda_{\text{eff}}}\,{}_2F_1[-\tfrac{1}{2}, \tfrac{1}{2}+2\nu; \tfrac{1}{2}; -\sin^2(\mu r)]\,, \end{aligned}$$

where $\Lambda_{\text{eff}} = -6\mu^2\nu^2$, and ψ_0 is an integration constant. Keep in mind that the parameters of the solution should be that ϕ^2 is non-negative for all $r \in (-\infty, \infty)$.

Trapping of gravity

It is subtler than the Randall-Sundrum case; but we believe that it traps gravity. The $D = 4$ on-shell graviton is

$$h_{\mu\nu} = \xi_{\mu\nu} e^{\mathrm{i} p\cdot x}\,, \qquad p^2 = 0 = p\cdot\xi = \xi^\mu_\mu\,.$$

$$\int dr\sqrt{-g}\,|\Psi_0|^2 = \int_{-\infty}^{\infty} dr\,[\cosh(\mu r)]^{-4\nu}\,.$$

Also substitute the background into the action, we have

$$\begin{aligned}\int_{-\infty}^{\infty} dr\sqrt{-g}L \;&\sim\; \int_{-\infty}^{\infty} \frac{\mu^2\nu(5\nu\cosh(2\mu r) - 5\nu - 4)}{[\cosh(\mu r)]^{2+4\nu}} \\ &=\; \tfrac{6\mu\nu}{2\nu+1}\Big({}_2F_1[1,-2\nu-1;2\nu+1;-1]-1\Big)\,.\end{aligned}$$

Both are finite for positive ν.

Conclusions

- Lovelock gravities can be generalized to include a generic non-minimally coupled p-form.
- The construction provides toy models of higher-derivative effective theories of strings.
- Theories contain rich structures in the solution space.
- In higher-derivative gravities with non-minimally coupled matter, one cannot read off the matter energy-momentum tensor by simply calculating the Einstein curvature tensor $G_{\mu\nu}$, and hence many no-go theorem can be evaded.
- Unusual black hole thermodynamics arises.
- QSR may be generally invalid.
- What we have done so far is only the tip of the iceberg, and there is no reason to rule them out.
- Quantum gravity remains elusive until one understands the vast space of possible theories that respect the underlying symmetry principle of General Relativity.